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UNSAFE AND SAFE BOUNDARIES OF THE STABILITY REGION OF SYSTEMS WITH DELAY IN THE CASE OF A PAIR OF PURE IMAGINARY ROOTS AND ONE ZERO ROOT†

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A criterion of unsafe and safe parts of the boundary of stability regions of the equilibrium states of systems with delay, when the characteristic equation has a pair of pure imaginary roots and one zero root, is given, which develops results obtained previously in [1–6]. © 2003 Elsevier Science Ltd. All rights reserved.

The problem of determining the unsafe and safe boundaries of the stability of equilibrium states of systems with delay was considered previously in [1–11]. Methods and algorithms for investigating the stability of systems with delay in critical cases, based on reducing them to truncated systems without delay were given in [7–11]. Formulae were obtained in [1–4] for a quantity, similar to the first Lyapunov quantity, for first-order and second-order equations with delay, in the case of second-order and arbitrary-order systems with delay. A criterion of safe and unsafe boundaries of the stability region was derived in [5] for systems with delay in one of the simplest cases of a pair of pure imaginary roots and one zero root. However, in these investigations, sufficiently simple and convenient criteria of unsafe and safe boundaries of the stability region of equilibrium states of systems with delay were not obtained in the general case of a pair of pure imaginary roots and one zero root, as occur for systems without delay [12].

Consider a system with delay, described by the equation

$$\ddot{x} = a_1x + b_1x(t-\tau) + a_2\dot{x} + b_2\dot{x}(t-\tau) + f(x, \dot{x}, x(t-\tau), \dot{x}(t-\tau)) \tag{1}$$

where a_1, a_2, b_1 and b_2 are constants and $\tau > 0$.

We will assume that the analytic or fairly continuous function $f(x_1, x_2, x_3, x_4)$ can be expanded in series

$$f = \sum_{1\le i\le k\le 4} a_{ik}x_ix_k + \sum_{1\le i\le k\le p\le 4} a_{ikp}x_ix_kx_p + \dots$$

where a_{ik} and a_{ikp} are constant coefficients.

We will assume that the characteristic equation

$$\Delta(p) = \begin{vmatrix} p & -1 \\ -a_1 - b_1e^{-p\tau} & p - a_2 - b_2e^{-p\tau} \end{vmatrix} = 0 \tag{2}$$

has roots $p_1 = 0, p_{2,3} = \pm i\omega$ and roots p_j, which satisfy the condition $\operatorname{Re}p_j < -\sigma < 0$, which is only possible if the following conditions are satisfied

$$\begin{aligned} a_1 + b_1 &= 0 \\ -\omega^2 - a_1 - b_1\cos(\omega\tau) - \omega b_2\sin(\omega\tau) &= 0 \\ -\omega a_2 + b_1\sin(\omega\tau) - \omega b_2\cos(\omega\tau) &= 0 \end{aligned} \tag{3}$$

with respect to the coefficients a_1, b_1, a_2 and b_2 of the linear part of Eq. (1), which occur in Eq. (2).

Following the method described previously in [7, 8], we will write the following third-order truncated system without delay for Eq. (1)

$$\dot{y}_j = p_jy_j + \Phi(y_1, y_2, y_3), \quad j = 1, 2, 3 \tag{4}$$

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where

$$\Phi = g_1 y_1^2 + B_{12}y_1y_2 + B_{23}y_2\bar{y}_2 + B_{13}y_1\bar{y}_2 + B_{22}y_2^2 + B_{33}\bar{y}_2^2 + g_2 y_1^3$$

y_1 is a real variable, and y_2 and $\bar{y}_2 = y_3$ are complex-conjugate variables.

Here

$$g_1 = \sum_{1 \le i \le k \le 4} a_{ik}\alpha_{i1}\alpha_{k1}, \quad g_2 = \sum_{1 \le i \le k \le p \le 4} a_{ikp}\alpha_{i1}\alpha_{k1}\alpha_{p1}$$

$$B_{rq} = \sum_{1 \le i \le k \le 4} a_{ik}(\alpha_{ir}\alpha_{kq} + \alpha_{iq}\alpha_{kr})(r = 1, q = 2;\ r = 1, q = 3;\ r = 2, q = 3) \tag{5}$$

$$B_{rr} = \sum_{1 \le i \le k \le 4} a_{ik}\alpha_{ir}\alpha_{kr}, \quad r = 2, 3$$

$$\alpha_{ij} = \Delta_{2i}(p_j)/\Delta'(p_j), \quad 1 \le i \le 2, \quad 1 \le j \le 3 \tag{6}$$

$$\alpha_{ij} = \exp(-p_j\tau)\alpha_{i-2,j}, \quad 3 \le i \le 4, \quad 1 \le j \le 3 \tag{7}$$

$$\Delta'(p_j) = 2p_j + \exp(-p_j\tau)(\tau b_1 + \tau p_j b_2 - b_2) - a_2 \tag{8}$$

and $\Delta_{2i}(p_j)$ are the cofactors of the elements of the second row and the ith column of the determinants $\Delta(p_j)$.

If system (4) has an unsafe (safe) boundary, the system described by Eq. (1) also has an unsafe (safe) boundary [7, 8].

After reduction to normal form [13], system (4) takes the form

$$\dot{u} = g_1 u^2 + B_{23}z\bar{z} + g_2^* u^3, \quad \dot{z} = i\omega z + B_{12}uz \tag{9}$$

where

$$g_2^* = g_2 + g_1(B_{13} - B_{12})/(i\omega) \tag{10}$$

The quantities g_1, g_2, B_{12}, B_{13} and B_{23} are given by formulae (5).

It follows from relations (9) and (10) that [5, 12], if $g_1 \ne 0$, the boundary of the stability region of the system described by Eq. (9) is unsafe. It is also unsafe if $g_1 = 0$, but either $g_2 > 0$ or $B_{23}\mathrm{Re}B_{12} > 0$, and is safe if $g_1 = 0$, $g_2 < 0$ and $B_{23}\mathrm{Re}B_{12} < 0$.

Example. Consider the equation

$$\ddot{x} = -\frac{\pi}{2}\dot{x}(t-1) + a_{11}x^2 + a_{12}x\dot{x} + a_{13}xx(t-1) + a_{14}x\dot{x}(t-1) + a_{111}x^3 \tag{11}$$

It has the form of Eq. (1).

The characteristic equation has the roots $p_1 = 0$, $p_{2,3} = \pm \pi/2$, $p_j(j = 4, 5, \ldots)$, which satisfy the inequality $\mathrm{Re}p_j < -\sigma < 0$ [10].

Since, for Eq. (11), $a_1 = a_2 = b_1 = 0$ and $b_2 = -\pi/2$ and $\omega = \pi/2$, it can be shown by a simple check that conditions (3) are satisfied for Eq. (11).

The quantity $\Delta'(p_j) = 2p_j + (\pi/2)\exp(-p_j)(1 - p_j)$.

We will obtain g_1 for Eq. (11). We have from formulae (5)–(8)

$$g_1 = (a_{11} + a_{13})/s^2, \quad s = \pi/2$$

If $g_1 \ne 0$, the boundary of the stability region of equilibrium state $x = 0$ of Eq. (11) is unsafe [5].

Consider the case when

$$g_1 = (a_{11} + a_{13})/s^2 = a_{12} = a_{14} = 0, \quad a_{11} \ne 0$$

In this case the nature of the boundary of the stability region of the equilibrium state $x = 0$ of Eq. (11), as was shown above, is determined by the signs of the quantities $R = B_{23}\mathrm{Re}B_{12}$ and g_2.

From (5)–(8) we obtain

$$B_{12} = a_{11}\frac{s^2 - s - i(2 + s^2 - s)}{s^2(s^2+1)}, \quad \mathrm{Re}B_{12} = a_{11}\frac{s^2 - s}{s^2(s^2+1)}, \quad B_{23} = 2a_{11}\frac{1}{s^2(s^2+1)}$$

We now conclude that

$$R = B_{23}\mathrm{Re}B_{12} = 2a_{11}^2\frac{s^2 - s}{s^4(s^2+1)} > 0$$

From (5) for Eq. (11) we obtain that

$$g_2 = a_{111}\alpha_{11}^3 = a_{111}/s^3$$

It follows from results obtained earlier in [12] that in the case when $g_1 = 0, R > 0$ and $a_{111} \neq 0$, the boundary of the stability region of the equilibrium state $x = 0$ of Eq. (11) is unsafe.

We will now assume that

$$a_{11} + a_{13} = 0, \quad a_{12} \neq 0, \quad a_{14} \neq 0$$

For Eq. (11) we have from formulae (5)

$$B_{rq} = 2a_{11}\alpha_{11}\alpha_{1q} + a_{12}(\alpha_{1r}\alpha_{2q} + \alpha_{1q}\alpha_{2r}) + a_{13}(\alpha_{1r}\alpha_{3q} + \alpha_{3r}\alpha_{1q}) + a_{14}(\alpha_{1r}\alpha_{4q} + \alpha_{4r}\alpha_{1q})$$
$$r = 1, \quad q = 2; \quad r = 2, \quad q = 3 \qquad (12)$$

where, by formulae (6)–(8)

$$\alpha_{22} = \frac{1 - is}{s^2+1}, \quad \alpha_{23} = \frac{1 + is}{s^2+1}, \quad \alpha_{42} = \frac{-s - i}{s^2+1}, \quad \alpha_{43} = \frac{-s + i}{s^2+1} \qquad (13)$$

From expressions (12) and (13), taking into account the fact that $a_{11} + a_{13} = 0$, we obtain

$$\mathrm{Re}B_{12} = a_{11}\frac{s-1}{s(s^2+1)} + a_{12}\frac{1}{s(s^2+1)} - a_{14}\frac{1}{s^2+1}$$

$$B_{23} = \frac{2a_{11}}{s^2(s^2+1)} + 2a_{12}\frac{s+1}{s^2+1} + 2a_{14}\frac{1}{s(s^2+1)}$$

Depending on the value of the coefficients a_{11}, a_{12}, a_{13} and a_{14} in the case when $a_{11} + a_{13} = g_1 = 0$, the value of R may be greater or less than zero. For example, when

$$a_{11} = a_{12} = a_{13} = 0, \quad a_{14} \neq 0$$

we have

$$R = -2a_{14}^2/(s(s^2+1)^2) < 0$$

whence it follows that in this case the boundary of the region of stability of the equilibrium state $x = 0$ of Eq. (11) is safe when $a_{111} < 0$ and unsafe when $a_{111} > 0$.

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